ON THE SOLUTION OF A BOUNDARY VALUE PROBLEM

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PMM Vol.23, No.3, 1959, pp.534-539

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(Received 16 February 1959)

We shall solve the systems of equations

$$\Delta\Phi_1 + a_{11}\Phi_1 + a_{12}m_1 = 0, \qquad \Delta m_1 + \nu\Phi_1 + a_{22}m_1 = 0 \tag{1}$$

$$\Delta\Phi_2 + b_{11}\Phi_2 + b_{12}m_2 = 0, \qquad \Delta m_2 + b_{22}m_2 = 0 \tag{2}$$

$$\Delta\Phi_3 + c_{11}\Phi_3 + c_{12}m_3 = 0, \qquad \Delta m_3 + c_{22}m_3 = 0 \tag{3}$$

for the cylindircal region represented in the figure. Here Φ_i, m_i represent solutions in the i-zone, a_{ik}, b_{ik} are the given coefficients, and ν is the eigenvalue of the boundary value problem.

The conditions on the zone boundaries are given in the following form

$$\frac{\partial\Phi_i}{\partial z} = \frac{\partial m_i}{\partial z} = 0 \quad \text{for} \quad z = \pm H \tag{4}$$

$$\frac{\partial\Phi_2}{\partial r} = \frac{\partial m_2}{\partial r} = 0 \quad \text{for } r = R \tag{5}$$

$$\left.\begin{aligned} \Phi_1 &= \Phi_2, & \frac{\partial\Phi_1}{\partial r} &= \delta_2\frac{\partial\Phi_2}{\partial r} \\ m_1 &= \delta_1 m_2, & \frac{\partial m_1}{\partial r} &= \delta_1\delta_2\frac{\partial m_2}{\partial r} \end{aligned}\right\} \quad \text{on the boundary of zones } (r = r_0, \ -h < z < h) \tag{6}$$

$$\left.\begin{aligned} \Phi_1 &= \Phi_3, & \frac{\partial\Phi_1}{\partial z} &= \varepsilon_2\frac{\partial\Phi_3}{\partial z} \\ m_1 &= m_3, & \frac{\partial m_1}{\partial z} &= \varepsilon_2\frac{\partial m_3}{\partial z} \end{aligned}\right\} \quad \text{on the boundary of zones } (z = \pm h, \ 0 < r < r_0) \tag{7}$$

$$\left.\begin{aligned} \Phi_2 &= \Phi_3, & \frac{\partial\Phi_2}{\partial r} &= \gamma_2\frac{\partial\Phi_3}{\partial r} \\ m_2 &= \frac{1}{\delta_1}m_3, & \frac{\partial m_2}{\partial r} &= \gamma_2\gamma_1\frac{\partial m_3}{\partial r} \end{aligned}\right\} \quad \left(\gamma_1 = \frac{1}{\delta_1}\right) \text{ on the boundary of zones } (r = r_0, \ h < |z| < H) \tag{8}$$

We are interested in the smallest possible ν.

It is clear that such a solution is independent of the angle, moreover, because of symmetry, it is sufficient to restrict attention to the region D.

Introducing new functions Ψ_i, n_i such that

$$\Psi_i = A_{11}{}^i\Phi_i + A_{12}{}^i m_i, \qquad n_i = A_{21}{}^i\Phi_i + A_{22}{}^i m_i$$

and choosing the coefficients A_{lk}^{i} accordingly, we shall bring the systems (1), (2), (3) to the form

$$\Delta\Psi_1 - \varkappa_1{}^2\Psi_1 = 0, \qquad \Delta n_1 - \sigma_1{}^2 n_1 = 0 \tag{1'}$$

$$\Delta\Psi_2 - \varkappa_2{}^2\Psi_2 = 0, \qquad \Delta n_2 - \sigma_2{}^2 n_2 = 0 \tag{2'}$$

$$\Delta\Psi_3 - \varkappa_3{}^2\Psi_3 = 0, \qquad \Delta n_3 - \sigma_3{}^2 n_3 = 0 \tag{3'}$$

The conditions (4) and (5) are preserved also for the new functions. With respect to the conditions (6)-(8), these will assume the form

$$\begin{aligned}
&\Psi_1 = \mu_{11}\Psi_2 + \mu_{12}n_2, && \frac{\partial\Psi_1}{\partial r} = \delta_2\left(\mu_{11}\frac{\partial\Psi_2}{\partial r} + \mu_{12}\frac{\partial n_2}{\partial r}\right) \\
&n_1 = \mu_{21}\Psi_2 + \mu_{22}n_2, && \frac{\partial n_1}{\partial r} = \delta_2\left(\mu_{21}\frac{\partial\Psi_2}{\partial r} + \mu_{22}\frac{\partial n_2}{\partial r}\right)
\end{aligned} \tag{6'}$$

$$\begin{aligned}
&\Psi_1 = \alpha_{11}\Psi_3 + \alpha_{12}n_3, && \frac{\partial\Psi_1}{\partial z} = \beta_{11}\frac{\partial\Psi_3}{\partial z} + \beta_{12}\frac{\partial n_3}{\partial z} \\
&n_1 = \alpha_{21}\Psi_3 + \alpha_{22}n_3, && \frac{\partial n_1}{\partial z} = \beta_{21}\frac{\partial\Psi_3}{\partial z} + \beta_{22}\frac{\partial n_3}{\partial z}
\end{aligned} \tag{7'}$$

$$\begin{aligned}
&\Psi_2 = \Psi_3 + \varepsilon_1 n_3, && \frac{\partial\Psi_2}{\partial r} = \gamma_2\left(\frac{\partial\Psi_3}{\partial r} + \varepsilon_1\frac{\partial n_3}{\partial r}\right) \\
&n_2 = n_3, && \frac{\partial n_2}{\partial r} = \gamma_2\frac{\partial n_3}{\partial r}
\end{aligned} \tag{8'}$$

where α_{ik}, β_{ik}, μ_{ik} are expressed in terms of δ_i, γ_i, ϵ_i and the coefficients of the equations. Without writing out these relations, we shall only note that $\beta_{ik} = \epsilon_2\alpha_{ik}$.

It is obvious that particular solutions of equations (2′), satisfying all conditions except those where $r = r_0$, will be the functions

$$\Psi_2^{(n)} = C_2^{(n)}\left[I_0\left(\sqrt{\varkappa_2^2 + \left(\frac{\pi n}{H}\right)^2} r\right)K_1\left(\sqrt{\varkappa_2^2 + \left(\frac{\pi n}{H}\right)^2} R\right) + \right.$$
$$\left. + I_1\left(\sqrt{\varkappa_2^2 + \left(\frac{\pi n}{H}\right)^2} R\right)K_0\left(\sqrt{\varkappa_2^2 + \left(\frac{\pi n}{H}\right)^2} r\right)\right]\cos\frac{\pi n}{H}z$$

$$n_2^{(n)} = C_3^{(n)}\left[I_0\left(\sqrt{\sigma_2^2 + \left(\frac{\pi n}{H}\right)^2} r\right)K_1\left(\sqrt{\sigma_2^2 + \left(\frac{\pi n}{H}\right)^2} R\right) + \right.$$
$$\left. + I_1\left(\sqrt{\sigma_2^2 + \left(\frac{\pi n}{H}\right)^2} R\right)K_0\left(\sqrt{\sigma_2^2 + \left(\frac{\pi n}{H}\right)^2} r\right)\right]\cos\frac{\pi n}{H}z, \quad n = 0, 1, 2, 3, \ldots$$

Let us formulate the problem: To construct in the zones *1-3* a system of eigenfunctions $Z(z)$, using the conditions when $z = 0$, $z = H$, $z = h$.

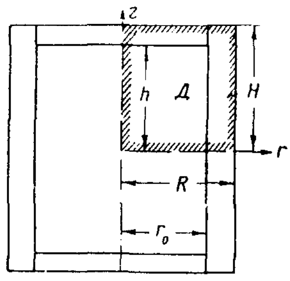

Fig. 1.

We shall regard $Z(z)$ as a vector with components Y and U, and we shall introduce the notations

$$Y = \begin{cases} y_1 & (0 \leqslant z < h), \\ y_3 & (h < z \leqslant H), \end{cases} \qquad U = \begin{cases} u_1 & (0 \leqslant z < h) \\ u_3 & (h < z \leqslant H) \end{cases}$$

It is clear that these functions are discontinuous when $z = h$, and that they satisfy the conditions

$$Z|_{h-0} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} Z|_{h+0}, \qquad Z'|_{h-0} = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} Z'|_{h+0} \tag{9}$$

and also the conditions

$$Z' = 0 \qquad \text{for} \quad z = 0, \quad z = H \tag{10}$$

In our notations the particular solutions of the equations (1′) and (3′), satisfying the conditions when $z = 0$, $z = H$ and $z = h$, will be written in the form

$$\begin{aligned} \Psi_1^{(k)} &= C_1^{(k)} I_0(\mu_k r)\, y_1^{(k)}(z), & n_1^{(k)} &= C_1^{(k)} I_0(\mu_k r)\, u_1^{(k)}(z) \\ \Psi_3^{(k)} &= C_1^{(k)} I_0(\mu_k r)\, y_3^{(k)}(z), & n_3^{(k)} &= C_1^{(k)} I_0(\mu_k r)\, u_3^{(k)}(z) \end{aligned} \tag{11}$$

where μ_k are the roots of the characteristic equation $\Lambda(\mu) = 0$, obtained from the condition (7′) by the obvious substitution

$$\cos \sqrt{\mu_k^2 - \varkappa_1^2}\, z, \quad \cos \sqrt{\mu_k^2 - \sigma_1^2}\, z, \quad \cos \sqrt{\mu_k^2 - \varkappa_3^2}\, (H - z), \quad \cos \sqrt{\mu_k^2 - \sigma_3^2}\, (H - z)$$

If the functions $Z(z)$ really satisfy the conditions claimed for the eigenfunctions, then we shall be able to construct a general solution both in zone *2* and in the zones *1-3*. Finally an application of the conditions when $r = r_0$ will permit us to determine the eigenvalue of the boundary value problem (to within a constant) and the unknown coefficients $C_1^{(k)}$, $C_2^{(n)}$, $C_3^{(n)}$.

We shall examine in greater detail the functions $Z(z)$. Let us introduce the operator

$$\Lambda = \begin{pmatrix} L & 0 \\ 0 & M \end{pmatrix}$$

The domain of definition of Λ is a set of vector functions Φ of argument z with continuous derivatives up to the second order, satisfying the conditions (9) and (10). Here, and from now on, the continuity of the derivatives of Φ is required everywhere in the region except at the point $z = h$. L and M are determined from the relations

$$LY = \begin{cases} Y'' - \varkappa_1^2 Y & (0 < z < h), \\ B(Y'' - \varkappa_3^2 Y) & (h < z < H), \end{cases} \qquad MU = \begin{cases} A(U'' - \sigma_1^2 U) & (0 < z < h) \\ C(U'' - \sigma_3^2 U) & (h < z < H) \end{cases} \tag{12}$$

A, B, C are for the time being undefined coefficients and Y and U components of the vector function. If we take for Y and U the components of the eigenfunction $Z(z)$, then clearly

$$LY = \begin{cases} -\mu^2 Y & (0 < z < h), \\ -B\mu^2 Y & (h < z < H), \end{cases} \qquad MU = \begin{cases} -A\mu^2 U & (0 < z < h) \\ -C\mu^2 U & (h < z < H) \end{cases} \tag{13}$$

We shall choose the constants A, B, C such that the operator Λ is selfconjugate, i.e. that the relation

$$(\Phi_1, \Lambda\Phi_2) = (\Lambda\Phi_1, \Phi_2) \tag{14}$$

is satisfied.

We will show how this can be done. Let us consider the expression

$$Q = \int_0^H [\Phi_1 \Lambda \Phi_2 - \Phi_2 \Lambda \Phi_1]\, dz \tag{15}$$

As before we will denote the components Φ_i by Y^i and U^i where

$$Y^i = \begin{cases} y_1^i & (0 \leqslant z < h), \\ y_3^i & (h < z \leqslant H) \end{cases} \qquad U^i = \begin{cases} u_1^i & (0 \leqslant z < h) \\ u_3^i & (h < z \leqslant H) \end{cases}$$

Let us transform the expression (15):

$$Q = \int_0^H [Y^{(1)} LY^{(2)} - Y^{(2)} LY^{(1)}]\, dz + \int_0^H [U^{(1)} MU^{(2)} - U^{(2)} MU^{(1)}]\, dz =$$

$$= (y_1^{(1)} y_1'^{(2)} - y_1'^{(1)} y_1^{(2)}) \Big|_0^h + B(y_3^{(1)} y_3'^{(2)} - y_3'^{(1)} y_3^{(2)}) \Big|_h^H +$$

$$+ A(u_1^{(1)} u_1'^{(2)} - u_1^{(2)} u_1'^{(1)}) \Big|_0^h + C(u_3^{(1)} u_3'^{(2)} - u_3^{(2)} u_3'^{(1)}) \Big|_h^H$$

Using the conditions for $z = 0$, $z = H$, $z = h$, we will get

$$Q = (\alpha_{11}y_3^{(1)} + \alpha_{12}u_3^{(1)})_{z=h}\, y_1'^{(2)}|_{z=h} - y_1^{(2)}|_{z=h}(\beta_{11}y_3'^{(1)} + \beta_{12}u_3'^{(1)})_{z=h} -$$
$$- By_3^{(1)}|_{z=h}y_3'^{(2)}|_{z=h} + \beta\, y'_3{}^{(2)}|_{z=h}y_3^{(1)}|_{z=h} + Au_1'^{(2)}|_{z=h}(\alpha_{21}y_3^{(1)} + \alpha_{22}u_3^{(1)})_{z=h} -$$
$$- Au_1^{(2)}(\beta_{21}y_3'^{(1)} + \beta_{22}u_3'^{(1)})|_{z=h} - Cu_3^{(1)}|_{z=h}u_3'^{(2)}|_{z=h} + Cu_3^{(2)}|_{z=h}u_3'^{(1)}|_{z=h}$$

We shall try to find values of A, B, C, for which Q will be zero. To this end the conditions

$$\alpha_{11}y_1'^{(2)} - By_3'^{(2)} + A\alpha_{21}u_1'^{(2)} = 0, \qquad -\beta_{11}y_1^{(2)} + By_3^{(2)} - A\beta_{21}u_1^{(2)} = 0 \quad (16)$$
$$\alpha_{12}y_1'^{(2)} + A\alpha_{22}u_1'^{(2)} - Cu_3'^{(2)} = 0, \qquad -\beta_{21}y_1^{(2)} - A\beta_{22}u_1^{(2)} + Cu_3^{(2)} = 0$$

or

$$y_1^{(2)} = \frac{\beta_{22}B}{\Delta\beta}\, y_3^{(2)} - \frac{\beta_{21}C}{\Delta\beta}\, u_3^{(2)} \qquad u_1^{(2)} = -\frac{\beta_{21}B}{\Delta\beta A}\, y_3^{(2)} + \frac{\beta_{11}C}{A\Delta\beta}\, u_3^{(2)} \quad (17)$$
$$y_1'^{(2)} = \frac{\alpha_{22}B}{\Delta\alpha}\, y_3'^{(2)} - \frac{C\alpha_{21}}{\Delta\alpha}\, u_3'^{(2)} \qquad u_1'^{(2)} = -\frac{\alpha_{21}B}{A\Delta\alpha}\, y_3'^{(2)} + \frac{\alpha_{11}C}{A\Delta\alpha}\, u_3'^{(2)}$$

must be satisfied for $z = h$, where

$$\Delta\alpha = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}, \qquad \Delta\beta = \beta_{11}\beta_{22} - \beta_{12}\beta_{21}$$

Since the functions $Y^{(2)}$, $U^{(2)}$ satisfy the same conditions as $Y^{(1)}$, $U^{(1)}$, we obtain the following relations among A, B, C:

$$\frac{\alpha_{22}B}{\Delta\alpha} = \beta_{11}, \qquad -\frac{C\alpha_{21}}{\Delta\alpha} = \beta_{12}, \qquad -\frac{\alpha_{12}B}{A\Delta\alpha} = \beta_{21}, \qquad \frac{\alpha_{11}C}{A\Delta\alpha} = \beta_{22} \quad (18)$$
$$\frac{\beta_{22}B}{\Delta\beta} = \alpha_{11}, \qquad \frac{-\beta_{21}C}{\Delta\beta} = \alpha_{12}, \qquad \frac{-\beta_{12}B}{\Delta\beta A} = \alpha_{21}, \qquad \frac{\beta_{11}C}{\Delta\beta A} = \alpha_{22} \quad (19)$$

The conditions (19) are consequences of (18) since

$$\beta_{ik} = \varepsilon_2\alpha_{ik}$$

From the first three relations of (18) we will find the unknowns A, B, C, and obtain

$$B = \frac{\beta_{11}}{\alpha_{22}}\,\Delta\alpha, \qquad C = -\frac{\beta_{12}}{\alpha_{21}}\,\Delta\alpha, \qquad A = -\frac{\alpha_{12}}{\beta_{21}}\,\frac{\beta_{11}}{\alpha_{22}} \quad (20)$$

Substituting the found A, B, C, in the last of the relations (18), we see that it is fulfilled automatically, since

$$\beta_{ik} = \varepsilon_2\alpha_{ik}$$

Thus, for A, B, C, computed by the formulas (20), the expression (15) vanishes.

If we take for the functions Φ_1 and Φ_2, Z_1 and Z_2, that is solutions of the equation $\Delta Z = -\mu^2 Z$ corresponding to two distinct eigenvalues, then keeping in mind that

$$LY^i = \begin{cases} -\mu^2 y_1^{\,i} & (0 < z < h), \\ -B\mu^2 y_3^{\,i} & (h < z < H), \end{cases} \qquad MU^i = \begin{cases} -A\mu^2 u_1^{\,i} & (0 < z < h) \\ -C\mu^2 u_3^{\,i} & (h < z < H) \end{cases}$$

we will obtain

$$\int_0^h y_1^{(1)} y_1^{(2)} dz + B \int_h^H y_3^{(1)} y_3^{(2)} dz + A \int_0^h u_1^{(1)} u_1^{(2)} dz + C \int_h^H u_3^{(1)} u_3^{(2)} dz = 0 \qquad (21)$$

Thus the eigenfunctions Z_1 and Z_2 turn out to be orthogonal in the sense of (21).

For the proof of other properties of the eigenfunctions we will use the theory of integral equations. To this end we shall construct Green's function $G(z, z_0)$. Such a function will be clearly a tensor of rank two:

$$G(z,\ z_0) = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \qquad (22)$$

As it is known, the eigenfunctions are determined by the relation

$$Z = -\mu^2 \int_0^H G(z,\ \zeta)\, Z(\zeta)\, d\zeta \qquad (23)$$

The function G satisfies the equation

$$\Lambda G = \begin{pmatrix} \delta(z - z_0) & 0 \\ 0 & \delta(z - z_0) \end{pmatrix} \qquad (24)$$

The components G_{12} and G_{21} are continuous at the point $z = z_0$ together with their first derivatives, and the components G_{11} and G_{12}, while themselves continuous at the point $z = z_0$, have a discontinuity in their derivatives.

$$\frac{dG_{11}}{dz}(z_0 + 0,\ z_0) - \frac{dG_{11}}{dz}(z_0 - 0,\ z_0) = \begin{cases} 1 & \text{for } z_0 < h \\ \frac{1}{B} & \text{for } z_0 > h \end{cases} \qquad (25)$$

$$\frac{dG_{22}}{dz}(z_0 + 0,\ z_0) - \frac{dG_{22}}{dz}(z_0 - 0,\ z_0) = \begin{cases} \frac{1}{A} & \text{for } z_0 < h \\ \frac{1}{C} & \text{for } z_0 > h \end{cases} \qquad (26)$$

Further, the function G must satisfy the boundary conditions

$$\frac{dG}{dz} = 0 \quad \text{for } z = 0 \text{ и } z = H \qquad (27)$$

$$G|_{h-0} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} G|_{h+0}, \qquad \left.\frac{dG}{dz}\right|_{h-0} = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} \left.\frac{dG}{dz}\right|_{h+0} \qquad (28)$$

As it is known, the solution of the equation

$$\Lambda\Phi = F$$

(where Φ is a vector with components Y, U and F is a vector with components f_1, f_2) is represented as:

$$\Phi = \int_0^H G(z, \zeta) F(\zeta)\, d\zeta, \quad Y = \int_0^H (G_{11}f_1 + G_{12}f_2)\, d\zeta, \quad U = \int_0^H (G_{21}f_1 + G_{22}f_2)\, d\zeta$$

Clearly

$$LY = f_1, \qquad MU = f_2 \tag{29}$$

Let us further apply to the integral equations under consideration the general theory of equations with a symmetrical root. To this end we shall prove the symmetry of Green s function, i.e. we will establish that

$$(\Phi_2,\ A\Phi_1) = (A\Phi_2,\ \Phi_1) \tag{30}$$

where Φ_1, Φ_2 are two vector functions, and the operator A is determined from the following relations

$$\begin{aligned} AY(z_1) &= \int_0^H [G_{11}(z_1,\ z_2) Y(z_2) + G_{12}(z_1,\ z_2) U(z_2)]\, dz_2 \\ AU(z_1) &= \int_0^H [G_{21}(z_1,\ z_2) Y(z_2) + G_{22}(z_1,\ z_2) U(z_2)]\, dz_2 \end{aligned} \tag{31}$$

Thus, the proof of formula (30) is reduced to the proof of the following relations

$$G_{11}(z_1, z_2) = G_{11}(z_2, z_1), \quad G_{12}(z_1, z_2) = G_{21}(z_2, z_1), \quad G_{22}(z_1, z_2) = G_{22}(z_2, z_1) \tag{32}$$

Let us establish the formulas (32) for Green's function, introduced according to (22). As shown above (see formula (15)) the relation

$$\int_0^H (\Phi_1\Lambda\Phi_2 - \Phi_2\Lambda\Phi_1)\, dz = 0$$

is valid for the vector functions Φ_1, and Φ_2, satisfying the equations

$$\Lambda\Phi_1 = F_1, \qquad \Lambda\Phi_2 = F_2$$

and the boundary conditions (9) and (10).

Let us take for the functions Φ_1 and Φ_2 vectors, which are a part of the tensors $G(z,\ z_1) = G^{(1)}$ and $G(z,\ z_2) = G^{(2)}$ Clearly

$$\Lambda G^{(1)} = \begin{pmatrix} \delta(z - z_1) & 0 \\ 0 & \delta(z - z_1) \end{pmatrix}, \qquad \Lambda G^{(2)} = \begin{pmatrix} \delta(z - z_2) & 0 \\ 0 & \delta(z - z_2) \end{pmatrix} \tag{33}$$

Applying formula (15) to the vectors

$$\begin{pmatrix} G_{11}^{(1)} \\ G_{21}^{(1)} \end{pmatrix} \text{ и } \begin{pmatrix} G_{11}^{(2)} \\ G_{21}^{(2)} \end{pmatrix}, \qquad \begin{pmatrix} G_{11}^{(1)} \\ G_{21}^{(1)} \end{pmatrix} \text{ и } \begin{pmatrix} G_{12}^{(2)} \\ G_{22}^{(2)} \end{pmatrix}$$

$$\begin{pmatrix} G_{12}^{(1)} \\ G_{22}^{(1)} \end{pmatrix} \text{ и } \begin{pmatrix} G_{11}^{(2)} \\ G_{21}^{(2)} \end{pmatrix}, \qquad \begin{pmatrix} G_{12}^{(1)} \\ G_{22}^{(1)} \end{pmatrix} \text{ и } \begin{pmatrix} G_{12}^{(2)} \\ G_{22}^{(2)} \end{pmatrix}$$

we obtain the formulas (32).

The symmetry of Green's function allows us to conclude immediately that all eigenvalues of our problem $\lambda = -\mu^2$ are real, and that the eigenfunctions form orthogonal systems in the sense of (21), as has already been established.

We shall now prove that the eigenfunctions form a complete system. To this end we shall use the Hilbert-Schmidt theory. It is sufficient to show that any twice continuously differentiable function satisfying the boundary conditions can be represented as a source by the root, i.e. that any vector Φ can be represented in the form of the integral

$$\Phi(z) = \int_0^H G(z, \zeta)\, h(\zeta)\, d\zeta \tag{34}$$

In order to obtain such a representation, we shall substitute the vector Φ in the left member of the equation

$$\Lambda\Phi = h \tag{35}$$

We will get the function h. It can easily be verified that $\lambda = 0$ is not the eigenvalue of our problem. Therefore, the solution of the equation (35) is unique, and is clearly expressed by the formula (34), which is what had to be proved. After the construction of the system of eigenfunctions $Z(z)$, a solution of the boundary value problem is easily obtained in the form of a series.

The authors have solved several versions of the afore-mentioned boundary value problem, where it was sufficient to restrict oneself to six terms for the functions $\Psi_2^{(n)}$, $n_2^{(2)}$, and nine terms for the functions $\Psi_1^{(k)}$, $\Psi_3^{(k)}$ and $n_1^{(k)}$, $n_3^{(k)}$. In the course of the solution of the problem an electronic computer BESM was used.

Translated by I.S.